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Generating varieties, Bott periodicity and instantons

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Let G be the classical group and let $\mathcal{M}_k(G)$ be the based moduli space of G-instantons on S^4 with instanton number k. It is known that $\mathcal{M}_k(G)$ yields real and symplectic Bott periodicity, however an explicit geometric description of the homotopy equivalence has not been known. We consider certain orbit spaces in $\mathcal{M}_k(G)$ and show that the restriction of the inclusion of $\mathcal{M}_k(G)$ into the moduli space of connections, which, in turn, is explicitly described by the commutator map of G. We prove this restriction satisfies a triple loop space version of the generating variety argument of Bott (1958) [5], and it also gives real and symplectic Bott periodicity. This also gives a new proof of real and symplectic Bott periodicity.

1. Introduction

Let G be a compact connected simple Lie group. Then there is an isomorphism $\pi_3(G) \cong \pi_4(BG) \cong \mathbb{Z}$. We will fix an isomorphism $\pi_3(G) \cong \mathbb{Z}$. Then principal G-bundles over S^4 are classified by $\mathbb{Z} = \pi_3(G)$, and denote by P_k the principal G-bundle over S^4 corresponding to $k \in \mathbb{Z}$. Let $\mathcal{C}_k(G)$ be the based moduli space of connections on P_k. Then we have a natural homotopy equivalence

$$\mathcal{C}_k(G) \simeq \Omega^3_k G$$

where $\Omega^3_k G$ stands for the path component of $\Omega^3 G$ corresponding to $k \in \mathbb{Z} = \pi_3(G)$. We will identify $\mathcal{C}_k(G)$ with $\Omega^3_k G$ by this homotopy equivalence. Let $\mathcal{M}_k(G)$ be the based moduli space of instantons on P_k. Then we have a map

$$\theta_k : \mathcal{M}_k(G) \to \Omega^3_0 G$$

defined by the composite of the inclusion $\mathcal{M}_k(G) \to \Omega^3_k(G) \simeq \mathcal{C}_k(G)$ and the homotopy equivalence $\Omega^3_k G \simeq \Omega_0 G$, the shift by $-k \in \mathbb{Z} = \pi_3(G)$.

The topology of the map θ_k was first studied by Atiyah and Jones [3], and, later, it was proved by Boyer, Hurtubise, Mann and Milgram [8], Kirwan [14] and Tian [18] that the map θ_k is a homotopy equivalence in a range, which is known as the Atiyah–Jones theorem. As a consequence of this result, Tian [18] showed that the colimit of the map θ_k yields real and symplectic Bott periodicity. However, an explicit geometric description of the homotopy equivalence is not known while Bott periodicity was given by a map explicitly defined by the commutator maps of the classical groups [6]. In [9], it is shown that the map θ_k has some relation with the commutator map of G when $k = 1$. Recall that Bott [5] also used

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the commutator maps to study the topology of loop spaces of Lie groups. Exploiting the above result of [9] in connection with the classical result of Bott [5], Kamiyama [12] studied a triple loop space analogue of generating varieties of Bott [5].

We will give a mild generalization of the above result of [9] for arbitrary k. Using this, we prove triple loop space version of the generating variety argument [5] in a sense somewhat different from [12], and also prove Bott periodicity. This yields a new proof of real and symplectic Bott periodicity. We will give applications of this result to the homotopy types of $\mathcal{M}_k(G)$.

2. Subgroups of classical groups isomorphic with SU(2)

Let G be a compact, connected, simple Lie group with a fixed isomorphism $\pi_3(G) \cong \mathbb{Z}$. Note that G acts on $\mathcal{M}_k(G)$ via the action of the basepoint free gauge group of P_k on $\mathcal{M}_k(G)$. As is shown in [9], there is an orbit of this action for $k = 1$ such that the restriction of $\theta_1 : \mathcal{M}_1(G) \to \Omega_0^3 G$ is presented by the commutator map of G. By putting additional assumption, we can prove this for arbitrary k by essentially the same way in [9] as follows.

Lemma 1. *Suppose that there exists a subgroup H of G isomorphic to $\mathrm{SU}(2) \approx S^3$ such that the inclusion $\iota : H \hookrightarrow G$ represents $k \in \mathbb{Z} = \pi_3(G)$. Then there exists $\omega \in \mathcal{M}_k(G)$ satisfying*:

(1) *The orbit space $G \cdot \omega$ is homeomorphic with $G/C(H)$, where $C(H)$ stands for the centralizer of H.*
(2) *Let Γ denote the composite*:

$$G/C(H) \approx G \cdot \omega \hookrightarrow \mathcal{M}_k(G) \xrightarrow{\theta_k} \Omega_0^3 G.$$

Then we have

$$\Gamma\bigl(gC(H)\bigr) \simeq g\iota(h)g^{-1}\iota(h)^{-1}$$

for $g \in G$, $h \in H$.

Proof. Let α be an asymptotically flat connection on P_k. We regard S^4 as $\mathbb{R}^4 \cup \{\infty\}$. Recall from [3] that the homotopy equivalence $\mathcal{C}_k(G) \xrightarrow{\simeq} \Omega_0^3 G$ takes $\alpha \in \mathcal{M}_k(G)$ into its 'pure gauge' $\hat{\alpha} : S^3 \to G$ at $\infty \in S^4$ normalized as $\hat{\alpha}(*) = e$, where $*$ and e are the basepoint of S^3 and unity of G, respectively. (See [3].) The action of the basepoint free gauge group of P_k is locally the conjugation by G. Then the map θ_k is G-equivariant under the action of G on $\Omega_0^3 G$ given by $g \cdot \lambda(x) = g\lambda(x)g^{-1}$ for $g \in G$, $\lambda \in \Omega_0^3 G$, $x \in S^3$.

Let P be a principal SU(2)-bundle over S^4 represented by $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$. In [2], an asymptotically flat instanton ϖ whose pure gauge represents $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(3))$. Then the proof is completed by putting ω to be the push forward of ϖ by the inclusion $\iota : H \cong \mathrm{SU}(2) \to G$. $\square$

The original form of Bott periodicity [6] is given by such a map Γ in Lemma 1 where $\mathrm{SU}(2) \approx S^3$ is replaced with $\mathrm{U}(1) \approx S^1$. On the other hand, there is known a deep relation between $\mathcal{M}_k(G)$ and Bott periodicity as in [14,17,18]. Then we expect the map Γ in Lemma 1 may yield real and symplectic Bott periodicity which has period 4. Also we expect $G/C(H)$ and Γ in Lemma 1 may yield a 3-fold loop analogue of a generating variety for a loop space of a Lie group, which is already studied by Kamiyama [12] in a slightly different sense, that is, algebras over the Kudo–Araki operations. Then we introduce a family of subgroups of the classical groups which are isomorphic with SU(2) by which we can prove the above argument.

Hereafter, we put $(\mathbf{G}, \mathbf{H}, d) = (\mathrm{Sp}, \mathrm{O}, 1), (\mathrm{SU}, \mathrm{U}, 2), (\mathrm{SO}, \mathrm{Sp}, 4)$. We will define a family of subgroups $S_{k,l}(\mathbf{G})$ of $\mathbf{G}(dk + l)$ indexed by positive integers k and non-negative integers l. Since the Lie group $\mathbf{G}(dk + l)$ must be simple, we will assume $dk + l > 4$ when $\mathbf{G} = \mathrm{SO}$.

Let $\mathbf{c} : \mathrm{O}(n) \to \mathrm{U}(n)$, $\mathbf{q} : \mathrm{U}(n) \to \mathrm{Sp}(n)$, $\mathbf{c}' : \mathrm{Sp}(n) \to \mathrm{SU}(2n)$, and $\mathbf{r} : \mathrm{U}(n) \to \mathrm{O}(2n)$ be the canonical inclusions. In order to make things clear, we write the maps $\mathbf{c}'$ and $\mathbf{r}$ explicitly as follows. Let $\mathrm{M}_n(\mathbb{K})$ be the set of all square matrices of order n over a field $\mathbb{K}$. For $A = (a_{ij})$, $B = (b_{ij}) \in \mathrm{M}_n(\mathbb{C})$ such that $A + B\mathbf{j} \in \mathrm{Sp}(n)$, we put

$$\mathbf{c}'(A + B\mathbf{j}) = \bigl(\mathbf{c}'(a_{ij} + b_{ij}\mathbf{j})\bigr)$$

where $\mathbf{c}'(a + \mathbf{j}b) = \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ for $a, b \in \mathbb{C}$. We also put, for $C = (c_{ij})$, $D = (d_{ij}) \in \mathrm{M}_n(\mathbb{R})$ such that $C + D\sqrt{-1} \in \mathrm{U}(n)$,

$$\mathbf{r}(C + D\sqrt{-1}) = \bigl(\mathbf{r}(c_{ij} + d_{ij}\sqrt{-1})\bigr)$$

where $\mathbf{r}(c + d\sqrt{-1}) = \left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$ for $c, d \in \mathbb{R}$. We denote the matrix $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$ by $A \oplus B$. We consider the following family of subgroups of the classical groups isomorphic with $\mathrm{SU}(2) \approx S^3$:

$$S_{k,l}(\mathrm{Sp}) = \{\alpha E_k \oplus E_l \in \mathrm{Sp}(k+l) \mid \alpha \in \mathrm{Sp}(1)\},$$
$$S_{k,l}(\mathrm{SU}) = \{A \oplus E_l \in \mathrm{SU}(2k+l) \mid A \in \mathbf{c}'(S_{k,0}(\mathrm{Sp}))\},$$
$$S_{k,l}(\mathrm{SO}) = \{B \oplus E_l \in \mathrm{SO}(4k+l) \mid B \in \mathbf{rc}'(S_{k,0}(\mathrm{Sp}))\}$$

where E_n is the identity matrix of order n. We easily see

$$\mathbf{c}'(S_{k,l}(\mathrm{Sp})) = S_{k,2l}(\mathrm{SU}), \qquad \mathbf{r}(S_{k,l}(\mathrm{SU})) = S_{k,2l}(\mathrm{SO}).$$

We fix an isomorphism $\pi_3(\mathbf{G}(dk+l)) \cong \mathbb{Z}$ such that the inclusion $S_{k,l} \to \mathbf{G}(dk+l)$ represents $k \in \mathbb{Z}$.

Let $C_{k,l}(\mathbf{G})$ denote the centralizer of $S_{k,l}(\mathbf{G})$ in $\mathbf{G}(dk+l)$. Then we have

$$C_{k,l}(\mathrm{Sp}) = \mathbf{qc}(\mathrm{O}(k)) \oplus \mathrm{Sp}(l).$$

We also denote by $C_{k,l}(\mathrm{U})$ the centralizer of $S_{k,l}(\mathrm{SU})$ in $\mathrm{U}(dk+l)$. Then we have

$$C_{k,l}(\mathrm{U}) = \{A \oplus B \in \mathrm{U}(2k+l) \mid A = (a_{ij}E_2) \in \mathrm{U}(2k),\ B \in \mathrm{U}(l)\}.$$

In order to describe the centralizer $C_{k,l}(\mathrm{SO})$, we give another description of $S_{k,l}(\mathrm{SO})$. Define the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{H}$ by

$$x \cdot (p,q) = p^{-1}xq$$

for $(p,q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $x \in \mathbb{H}$. It is well known that this action yields the universal covering homomorphism $\rho : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{Spin}(4) \to \mathrm{SO}(4)$. Then it easily follows that

$$S_{k,l}(\mathrm{SO}) = \{\underbrace{A \oplus \cdots \oplus A}_{k} \oplus E_l \mid A \in \rho(1 \times \mathrm{Sp}(1)) \subset \mathrm{SO}(4)\}.$$

We denote the extension $\mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ of $\rho|_{\mathrm{Sp}(1)\times 1}$ ambiguously by the same ρ. Then one can easily verify

$$\rho(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}) = \begin{pmatrix} x & y & z & w \\ -y & x & w & -z \\ -z & -w & x & y \\ -w & z & -y & x \end{pmatrix}$$

for $x, y, z, w \in \mathbb{R}$. The map $\rho : \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ induces a map $\bar{\rho} : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{4n}(\mathbb{R})$ by $\bar{\rho}(a_{ij}) = (\rho(a_{ij}))$ for $(a_{ij}) \in \mathrm{M}_n(\mathbb{H})$. Now we obtain

$$C_{k,l}(\mathrm{SO}) = \{\bar{\rho}(A) \oplus B \in \mathrm{SO}(4k+l) \mid A \in \mathrm{Sp}(k),\ B \in \mathrm{SO}(l)\}. \tag{1}$$

Summarizing the above observation on $\mathbb{C}_{k,l}(\mathbf{G})$, we get:

Proposition 2. *There are isomorphisms*

$$C_{k,l}(\mathrm{Sp}) \cong \mathrm{O}(k) \times \mathrm{Sp}(l),$$
$$C_{k,l}(\mathrm{U}) \cong \mathrm{U}(k) \times \mathrm{U}(l),$$
$$C_{k,l}(\mathrm{SO}) \cong \mathrm{Sp}(k) \times \mathrm{SO}(l)$$

satisfying a commutative diagram:

$$\begin{tikzcd} C_{k,l}(\mathrm{Sp}) \arrow[r, "\mathbf{c}'"] \arrow[d, "\cong"] & C_{k,2l}(\mathrm{U}) \arrow[r, "\mathbf{r}"] \arrow[d, "\cong"] & C_{k,4l}(\mathrm{SO}) \arrow[d, "\cong"] \\ \mathrm{O}(k) \times \mathrm{Sp}(l) \arrow[r, "\mathbf{c}\times\mathbf{c}'"] & \mathrm{U}(k) \times \mathrm{U}(2l) \arrow[r, "\mathbf{q}\times\mathbf{r}"] & \mathrm{Sp}(k) \times \mathrm{SO}(4l). \end{tikzcd}$$

We now define a space and a map corresponding to the orbit space and the map Γ in Lemma 1 with respect to $S_{k,l}(\mathbf{G})$. We define a space $\mathcal{X}_{k,l}(\mathbf{G})$ by

$$\mathcal{X}_{k,l}(\mathbf{G}) = \mathbf{G}(dk+l)/C_{k,l}(\mathbf{G})$$

and a map $\Gamma_{k,l} : S_{k,l}(\mathbf{G}) \wedge \mathcal{X}_{k,l}(\mathbf{G}) \to \mathbf{G}(dk+l)$ by

$$\Gamma_{k,l}(s, gC_{k,l}(\mathbf{G})) = gsg^{-1}s^{-1}$$

for $s \in S_{k,l}(\mathbf{G})$, $g \in \mathbf{G}(dk+l)$. We will identify $S_{k,l}(\mathbf{G})$ with S^3 if there is no confusion. It is obvious that the inclusions $\mathbf{G}(dk+l) \to \mathbf{G}(dk+(l+1))$ and $\mathbf{G}(dk+l) \to \mathbf{G}(d(k+1)+l)$ induce the commutative diagram:

$$\begin{tikzcd}
S^3 \wedge \mathcal{X}_{k+1,l}(\mathbf{G}) \arrow[d, "\Gamma_{k+1,l}"] & S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G}) \arrow[l] \arrow[r] \arrow[d, "\Gamma_{k,l}"] & S^3 \wedge \mathcal{X}_{k,l+1}(\mathbf{G}) \arrow[d, "\Gamma_{k,l+1}"] \\
\mathbf{G}(d(k+1)+l) & \mathbf{G}(dk+l) \arrow[l] \arrow[r] & \mathbf{G}(dk+(l+1)).
\end{tikzcd} \tag{2}$$

By the above observation on $C_{k,l}(\mathrm{SU})$ and $C_{k,l}(\mathrm{U})$, we see that there is a diffeomorphism:

$$\mathcal{X}_{k,l}(\mathrm{SU}) \cong \mathrm{U}(2k+l)/C_{k,l}(\mathrm{U}). \tag{3}$$

Note that $\mathbf{c}' : \mathrm{Sp}(k+l) \to \mathrm{SU}(2k+2l)$ and $\mathbf{r} : \mathrm{SU}(k+l) \to \mathrm{SO}(2k+2l)$ are homomorphisms which restrict to surjections $S_{k,l}(\mathrm{Sp}) \to S_{k,2l}(\mathrm{SU})$ and $S_{k,l}(\mathrm{SU}) \to S_{k,2l}(\mathrm{SO})$, respectively. Then they induce maps $\mathbf{c}' : \mathcal{X}_{k,l}(\mathrm{Sp}) \to \mathcal{X}_{k,2l}(\mathrm{SU})$ and $\mathbf{r} : \mathcal{X}_{k,l}(\mathrm{SU}) \to \mathcal{X}_{k,2l}(\mathrm{SO})$ satisfying a commutative diagram:

$$\begin{tikzcd}
S^3 \wedge \mathcal{X}_{k,l}(\mathrm{Sp}) \arrow[r, "1\wedge\mathbf{c}'"] \arrow[d, "\Gamma_{k,l}"] & S^3 \wedge \mathcal{X}_{k,2l}(\mathrm{SU}) \arrow[r, "1\wedge\mathbf{r}"] \arrow[d, "\Gamma_{k,2l}"] & S^3 \wedge \mathcal{X}_{k,4l}(\mathrm{SO}) \arrow[d, "\Gamma_{k,2l}"] \\
\mathrm{Sp}(k+l) \arrow[r, "\mathbf{c}'"] & \mathrm{SU}(2k+2l) \arrow[r, "\mathbf{r}"] & \mathrm{SO}(4k+4l).
\end{tikzcd} \tag{4}$$

We observe a relation between $\mathcal{X}_{1,l}(\mathbf{G})$ and a projective space. It follows from Proposition 2 that $\mathcal{X}_{1,l}(\mathrm{Sp}) = \mathbb{R}P^{4l+3}$ and also that $\mathcal{X}_{1,l}(\mathrm{SU})$ is the total space of the unit tangent bundle of $\mathbb{C}P^{l+1}$. Note that the map $\rho : \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ above induces a homomorphism $\rho : \mathrm{Sp}(n) \to \mathrm{SO}(4n)$. Then there is a map $\mathbb{H}P^{[\frac{l}{4}]} \to \mathcal{X}_{1,l}(\mathrm{SO})$ which is natural with respect to the maps $\mathbb{H}P^{[\frac{l}{4}]} \to \mathbb{H}P^{[\frac{l+1}{4}]}$ and $\mathcal{X}_{1,l}(\mathrm{SO}) \to \mathcal{X}_{1,l+1}(\mathrm{SO})$. We regard $\mathbb{H}P^{[\frac{l}{4}]}$ to be a subspace of $\mathcal{X}_{1,l}(\mathrm{SO})$ by this map. Put $\Gamma'_{1,l}$ to be the restriction of $\Gamma_{1,l} : S^3 \wedge \mathcal{X}_{4,l}(\mathrm{SO}) \to \mathrm{SO}(4+l)$ onto $\mathbb{H}P^{[\frac{l}{4}]} \subset \mathcal{X}_{4,l}(\mathrm{SO})$. Then we have an obvious commutative diagram:

$$\begin{tikzcd}
S^3 \wedge \mathbb{H}P^{[\frac{l}{4}]} \arrow[r] \arrow[d, "\Gamma'_{1,l}"] & S^3 \wedge \mathbb{H}P^{[\frac{l+1}{4}]} \arrow[d, "\Gamma'_{1,l+1}"] \\
\mathrm{SO}(4+l) \arrow[r] & \mathrm{SO}(5+l).
\end{tikzcd} \tag{5}$$

We next consider the map $\Gamma_{k,l}$ when l tends to ∞. Put $\mathcal{X}_{k,\infty}(\mathbf{G}) = \operatorname{colim}_l \mathcal{X}_{k,l}(\mathbf{G})$. Then, by (2), we have a map

$$\operatorname*{colim}_l \Gamma_{k,l} : S^3 \wedge \mathcal{X}_{k,\infty}(\mathbf{G}) \to \mathbf{G}(\infty)$$

which we denote by $\Gamma_{k,\infty}$ for $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$, there is a principal bundle

$$\mathbf{H}(k) \to \mathbf{G}(dk+l)/\mathbf{G}(l) \to \mathcal{X}_{k,l}(\mathbf{G})$$

by Proposition 2 where $\mathbf{G}(dk+l)/\mathbf{G}(l)$ is $(4l+2)$-connected and $(l-1)$-connected according as $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$. By Proposition 2 and (3), we also have a principal bundle

$$\mathrm{U}(k) \to \mathrm{U}(2k+l)/\mathrm{U}(l) \to \mathcal{X}_{k,l}(\mathrm{SU})$$

in which $\mathrm{U}(2k+l)/\mathrm{U}(l)$ is $2l$-connected. Then it follows that there is a homotopy equivalence

$$\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$$

and thus we obtain a map

$$\Gamma_{k,\infty} : S^3 \wedge B\mathbf{H}(k) \to \mathbf{G}(\infty).$$

Moreover, by Proposition 2 and (4), we get:

Proposition 3. *There is a homotopy commutative diagram*:

$$\begin{tikzcd}
S^3 \wedge B\mathrm{O}(k) \arrow[r, "1\wedge\mathbf{c}"] \arrow[d, "\Gamma_{k,\infty}"] & S^3 \wedge B\mathrm{U}(k) \arrow[r, "1\wedge\mathbf{q}"] \arrow[d, "\Gamma_{k,\infty}"] & S^3 \wedge B\,\mathrm{Sp}(k) \arrow[d, "\Gamma_{k,\infty}"] \\
\mathrm{Sp}(\infty) \arrow[r, "\mathbf{c}'"] & \mathrm{SU}(\infty) \arrow[r, "\mathbf{r}"] & \mathrm{SO}(\infty).
\end{tikzcd}$$

Note that, by (5), we also have a map $\Gamma'_{1,\infty} : S^3 \wedge \mathbb{H}P^\infty \to \mathrm{SO}(\infty)$ which coincides with the map $\Gamma_{1,\infty} : S^3 \wedge B\,\mathrm{Sp}(1) \to \mathrm{SO}(\infty)$.

We see from (2) that $\Gamma_{k,\infty}$ satisfies a homotopy commutative diagram:

$$\begin{tikzcd} S^3 \wedge B\mathbf{H}(k) \arrow[r] \arrow[d, "\Gamma_{k,\infty}"] & S^3 \wedge B\mathbf{H}(k+1) \arrow[d, "\Gamma_{k+1,\infty}"] \\ \mathbf{G}(\infty) \arrow[r, equal] & \mathbf{G}(\infty) \end{tikzcd} \tag{6}$$

where the top horizontal arrow is induced from the inclusion $\mathbf{H}(k) \to \mathbf{H}(k+1)$. Then we get a map

$$\Gamma_{\infty,\infty} = \operatorname*{colim}_k \Gamma_{k,\infty} : S^3 \wedge B\mathbf{H}(\infty) \to \mathbf{G}(\infty).$$

Let $\mu : \mathbf{G}(n) \times \mathbf{G}(n) \to \mathbf{G}(2n)$ be an inclusion such as by $\mu(A, B) = A \oplus B$ for $A, B \in \mathbf{G}(n)$. Then μ induces a map $\mathcal{X}_{k,l}(\mathbf{G}) \times \mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{2k,2l}(\mathbf{G})$, denoted by the same symbol μ, which yields the standard H-space structure on $B\mathbf{H}(\infty) \simeq \mathcal{X}_{\infty,\infty}(\mathbf{G})$. Moreover, the map μ satisfies a commutative diagram:

$$\begin{tikzcd} S^3 \wedge (\mathcal{X}_{k,l}(\mathbf{G}) \times \mathcal{X}_{k,l}(\mathbf{G})) \arrow[r, "1\wedge\mu"] \arrow[d, "\Delta"'] & S^3 \wedge \mathcal{X}_{2k,2l}(\mathbf{G}) \arrow[dd, "\Gamma_{2k,2l}"] \\ (S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G})) \times (S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G})) \arrow[d, "\Gamma_{k,l}\times\Gamma_{k,l}"'] & \\ \mathbf{G}(dk+l) \times \mathbf{G}(dk+l) \arrow[r, "\mu"] & \mathbf{G}(2dk+2l) \end{tikzcd}$$

where Δ is defined by $\Delta(s, x, y) = (s, x, s, y)$ for $s \in S^3$, $x, y \in \mathcal{X}_{k,l}(\mathbf{G})$. Let $\mathrm{ad} : [\Sigma X, Y] \cong [X, \Omega Y]$ denote the adjoint congruence. Then we have established:

Lemma 4. *The map* $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ *is an H-map.*

We will show that the image of $\mathrm{ad}^3\,\Gamma_{1,l}$ in homology generates the Pontrjagin ring of $\Omega_0^3\mathbf{G}(dk+l)$ in a range, which is an analogue of the generating variety for a loop space of a Lie group, and that the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty}$ yields Bott periodicity.

3. Cohomology calculation for $\Gamma_{1,l}$

In this section, we give a cohomology calculation for the map $\Gamma_{1,l}$ and $\Gamma'_{1,l}$. We first consider the case $\mathbf{G} = \mathrm{SO}$. In this case, we calculate $\Gamma'_{1,l}$ in cohomology instead of $\Gamma_{1,l}$ since the cohomology of $\mathcal{X}_{1,l}(\mathrm{SO})$ is complicated as is seen in [13].

Proposition 5. *For $l \geqslant 4$, the map $(\Gamma'_{1,l})^* : H^*(\mathrm{SO}(4+l); \mathbb{Z}/2) \to H^*(S^3 \wedge \mathbb{H}P^{[\frac{l}{4}]}; \mathbb{Z}/2)$ is surjective.*

Proof. Recall first that the mod 2 cohomology of $\mathrm{SO}(4+l)$ is given as

$$H^*\big(\mathrm{SO}(4+l); \mathbb{Z}/2\big) = \mathbb{Z}/2[x_1, x_3, \ldots] \quad \text{for } * \leqslant 3+l,$$

where x_i is the suspension of the Stiefel–Whitney class w_{i+1}. Let u_3 be a generator of $H^3(S^3; \mathbb{Z}/2)$. Then, by definition, the inclusion $\iota : S^3 = S_{1,l}(\mathrm{SO}) \to \mathrm{SO}(4+l)$ induces the map in cohomology such as $\iota^*(x_3) = u_3$.

Let us consider the case $l = 12$. Let $P\,\mathrm{SO}(n)$ denote the n-dimensional projective orthogonal group, that is, $\mathrm{SO}(n)$ divided by its center. It is well known that

$$H^*\big(P\,\mathrm{SO}(16); \mathbb{Z}/2\big) = \mathbb{Z}/2[v, \bar{x}_1, \bar{x}_3, \bar{x}_5, \bar{x}_7] \quad \text{for } * \leqslant 7$$

where $|v| = 1$ and $\pi^*(\bar{x}_i) = x_i$ for the projection $\pi : \mathrm{SO}(16) \to P\,\mathrm{SO}(16)$. Moreover, we see from [4] that the Hopf algebra structure of $H^*(P\,\mathrm{SO}(16); \mathbb{Z}/2)$ is given as

$$\bar{\phi}^*(v) = 0, \qquad \bar{\phi}^*(\bar{x}_i) = \sum_{j=1}^{i} a_{ij}\bar{x}_j \otimes v^{i-j}$$

for $i = 1, 3, 5, 7$ in which $a_{53} = 0$, $a_{73} = 1$, where $\bar{\phi}$ stands for the reduced comultiplication. Let $\gamma : P\,\mathrm{SO}(16) \wedge P\,\mathrm{SO}(16) \to P\,\mathrm{SO}(16)$ be the reduced commutator map and let $\tilde{\gamma} : \mathrm{SO}(16) \wedge P\mathrm{O}(16) \to \mathrm{SO}(16)$ be a lift of γ. Then by a straightforward calculation, we have

$$\tilde{\gamma}^*(x_7) = u_3 \otimes v^4.$$

On the other hand, since the center of SO(16) is included in $C_{1,12}(\mathrm{SO})$, we have the projection $P\,\mathrm{SO}(16) \to \mathcal{X}_{1,12}(\mathrm{SO})$ satisfying a commutative diagram:

$$\begin{tikzcd}
\mathrm{SO}(16) \arrow[r] \arrow[d, "\pi"'] & \mathrm{SO}(16)/\mathrm{SO}(12) \arrow[d] & \mathrm{Sp}(4)/\mathrm{Sp}(3) \arrow[l, "\mathbf{rc}'"'] \arrow[d] \\
P\,\mathrm{SO}(16) \arrow[r] \arrow[d] & \mathcal{X}_{1,12}(\mathrm{SO}) \arrow[d] & \mathbb{H}P^3 \arrow[l] \arrow[d] \\
B(\mathbb{Z}/2) \arrow[r] & B\,\mathrm{Sp}(1) \arrow[r, equal] & B\,\mathrm{Sp}(1)
\end{tikzcd}$$

where $\mathbb{Z}/2$ is the center of Sp(1). Then we see that a generator x of $H^4(\mathcal{X}_{1,12}(\mathrm{SO}); \mathbb{Z}/2)$ satisfies

$$\pi^*(x) = v^4, \qquad (\mathbf{rc}')^*(x) = q,$$

where q is a generator of $H^4(\mathbb{H}P^n; \mathbb{Z}/2)$. Now we have a commutative diagram:

$$\begin{tikzcd}
S^3 \wedge P\,\mathrm{SO}(16) \arrow[r] \arrow[d, "\tilde{\gamma}"] & S^3 \wedge \mathcal{X}_{1,12}(\mathrm{SO}) \arrow[d, "\Gamma_{1,12}"] & S^3 \wedge \mathbb{H}P^3 \arrow[l] \arrow[d, "\Gamma'_{1,12}"] \\
\mathrm{SO}(16) \arrow[r, equal] & \mathrm{SO}(16) \arrow[r, equal] & \mathrm{SO}(16).
\end{tikzcd}$$

Then we obtain

$$(\Gamma'_{1,12})^*(x_7) = u_3 \otimes q.$$

By (5), we have established

$$(\Gamma'_{1,l})^*(x_7) = u_3 \otimes q. \tag{7}$$

By the Wu formula, we have

$$\mathrm{Sq}^4 x_{4i-1} = (i-1)x_{4i+3}, \qquad \mathrm{Sq}^8 x_{4i-1} = \binom{i-1}{2} x_{4i+7}$$

in $H^*(\mathrm{SO}(4+l); \mathbb{Z}/2)$ for $* < 4+l$. Then, applying this to (7), the proof is completed. □

Proposition 6. *For $i > 0$, the map $\Gamma^*_{1,l} : H^{4i+3}(\mathrm{Sp}(1+l); \mathbb{Z}/2) \to H^{4i+3}(S^3 \wedge \mathbb{R}P^{4l+3}; \mathbb{Z}/2)$ is surjective.*

Proof. Let w and q be generators of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ and $H^4(\mathbb{H}P^\infty; \mathbb{Z}/2)$, respectively. Then the map $\mathbf{qc} : \mathbb{R}P^\infty \to \mathbb{H}P^\infty$ induces $(\mathbf{qc})^*(q) = w^4$ in cohomology. Recall that the mod 2 cohomology of $\mathrm{Sp}(n)$ is given as

$$H^*(\mathrm{Sp}(n); \mathbb{Z}/2) = \Lambda(y_3, y_7, \ldots, y_{4n-1})$$

where y_{4i-1} is the suspension of the modulo 2 reduction of the symplectic Pontrjagin class q_i. Then we have $(\mathbf{rc}')^*(x_{4i-1}) = y_{4i-1}$ here we use the same notation for the mod 2 cohomology of $\mathrm{SO}(\infty)$ as in the proof of Proposition 5. Then, for $l = \infty$, the proposition follows from Proposition 5 and (3). Thus the proof is completed by (2). □

Let $X\langle n\rangle$ denote the n-connective cover of a path-connected space X. Then, in general, any map $f : S^3 \wedge A \to X$ with A path-connected lifts to $X\langle 3\rangle$ which we denote by $\tilde{f}$.

Proposition 7. *Any lift $\tilde{\Gamma}_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to (\mathrm{SU}(\infty))\langle 3\rangle$ of $\Gamma_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to \mathrm{SU}(\infty)$ induces an isomorphism $\tilde{\Gamma}^*_{1,\infty} : H^5((\mathrm{SU}(\infty))\langle 3\rangle; \mathbb{Z}) \xrightarrow{\cong} H^5(S^3 \wedge \mathbb{C}P^\infty; \mathbb{Z})$.*

Proof. We will denote the modulo p reduction in cohomology by ρ_p for a prime p.

The integral cohomology of $\mathrm{SU}(n)$ is

$$H^*\big(\mathrm{SU}(n);\mathbb{Z}\big)=\Lambda(e_3,e_5,\dots,e_{2n-1}),$$

where e_{2i-1} is the suspension of the Chern class c_i. Then, by considering the Serre spectral sequence of a fibre sequence $\mathbb{C}P^\infty\to(\mathrm{SU}(\infty))\langle 3\rangle\xrightarrow{q}\mathrm{SU}(\infty)$, we see that $H^5((\mathrm{SU}(\infty))\langle 3\rangle;\mathbb{Z})\cong\mathbb{Z}$ is generated by ϵ such that

$$q^*(e_5)=2\epsilon. \tag{8}$$

Let $P\,\mathrm{SU}(n)$ be the n-dimensional projective unitary group, that is, $\mathrm{SU}(n)$ divided by its center. Let p be an odd prime. In [4], it is shown that

$$H^*\big(P\,\mathrm{SU}\big(p^r\big);\mathbb{Z}/p\big)=\mathbb{Z}/p[v]/\big(v^{p^r}\big)\otimes\Lambda(\bar{e}_1,\bar{e}_3,\dots,\bar{e}_{2p^r-1})$$

where $|v|=2$ and $\bar{\pi}^*(\bar{e}_i)=\rho_p(e_i)$ for the projection $\bar{\pi}:\mathrm{SU}(p^r)\to P\,\mathrm{SU}(p^r)$. Moreover, for the reduced comultiplication $\bar{\phi}$, we have

$$\bar{\phi}(\bar{e}_5)=a_1\bar{e}_3\otimes v+a_2\bar{e}_1\otimes v^2$$

for $a_1,a_2\in(\mathbb{Z}/p)^\times$. Let c and u_3 be generators of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ and $H^3(S^3;\mathbb{Z})$ respectively. Then, as in the proof of Proposition 5, we see that

$$\Gamma^*_{1,\infty}\big(\rho_p(e_5)\big)=a\rho_p(u_3\otimes c)$$

for $a\in(\mathbb{Z}/p)^\times$. Note that the above equation holds for any odd prime p. Then we have obtained, in the integral cohomology, that

$$\Gamma^*_{1,\infty}(e_5)=\pm 2^b u_3\otimes c$$

for some non-negative integer b, and thus by (8),

$$\tilde{\Gamma}^*_{1,\infty}(\epsilon)=\pm 2^{b-1}u_3\otimes c$$

which implies that b is positive. Since $H^5(S^3\wedge\mathbb{R}P^2;\mathbb{Z})\cong\mathbb{Z}/2$, it follows from Lemma 9 below and (2) that $\tilde{\Gamma}_{1,\infty}(\rho_2(\epsilon))\neq 0$ in the mod 2 cohomology, which yields $b=1$. Thus the proof is done. □

Lemma 8. *Let $\theta:\mathbb{R}P^2\to\mathrm{SO}(6)$ and $\iota:S^3=S_{1,2}(\mathrm{SO})\to\mathrm{SO}(6)$ be the inclusions. Then the Samelson product $\langle\iota,\theta\rangle$ is essential.*

Proof. By the adjointness of Whitehead products and Samelson products, we show that the Whitehead product of $\mathrm{ad}^{-1}\iota:S^4\to B\,\mathrm{SO}(6)$ and $\mathrm{ad}^{-1}\theta:\Sigma\mathbb{R}P^2\to B\,\mathrm{SO}(6)$ is essential. Suppose now that $[\mathrm{ad}^{-1}\iota,\mathrm{ad}^{-1}\theta]=0$. Then there exists a map $\kappa:S^4\times\Sigma\mathbb{R}P^2\to B\,\mathrm{SO}(6)$ satisfying the homotopy commutative diagram:

$$\begin{tikzcd} S^4\vee\Sigma\mathbb{R}P^2 \arrow[r, "\mathrm{ad}^{-1}\iota\vee\mathrm{ad}^{-1}\theta"] \arrow[d] & B\,\mathrm{SO}(6) \arrow[d, equal] \\ S^4\times\Sigma\mathbb{R}P^2 \arrow[r, "\kappa"] & B\,\mathrm{SO}(6). \end{tikzcd}$$

Let w and u_4 be generators of $H^1(\mathbb{R}P^2;\mathbb{Z}/2)$ and $H^4(S^4;\mathbb{Z}/2)$, respectively. Then, by definition, we have $\kappa^*(w_3)=1\otimes\Sigma w^2$ and $\kappa^*(w_4)=u_4\otimes 1$, where w_i is the Stiefel–Whitney class. On the other hand, it follows from the Wu formula that $\mathrm{Sq}^3w_4=w_3w_4$. Thus we obtain

$$0=\mathrm{Sq}^3(u_4\otimes 1)=\mathrm{Sq}^3\kappa^*(w_4)=\kappa^*\big(\mathrm{Sq}^3w_4\big)=\kappa^*(w_3w_4)=u_4\otimes\Sigma w^2\neq 0$$

which is a contradiction. Therefore we have established the Whitehead product $[\mathrm{ad}^{-1}\iota,\mathrm{ad}^{-1}\theta]$ is essential. □

Recall that there is an isomorphism $\mathrm{SU}(4)\cong\mathrm{Spin}(6)$. Since the center of $\mathrm{SU}(4)\cong\mathrm{Spin}(6)$ is included in $C_{1,2}(\mathrm{SU})$, there is a projection $\pi:\mathrm{SO}(6)\to\mathcal{X}_{1,2}(\mathrm{SU})$.

Lemma 9. *Let $\theta:\mathbb{R}P^2\to\mathrm{SO}(6)$ be the inclusion and let $\lambda:S^3\wedge\mathbb{R}P^2\to(\mathrm{SU}(4))\langle 3\rangle$ be the composite:*

$$S^3\wedge\mathbb{R}P^2\xrightarrow{1\wedge\theta}S^3\wedge\mathrm{SO}(6)\xrightarrow{1\wedge\pi}S^3\wedge\mathcal{X}_{1,2}(\mathrm{SU})\xrightarrow{\tilde{\Gamma}_{1,2}}\big(\mathrm{SU}(4)\big)\langle 3\rangle.$$

Then $\lambda^*(\epsilon)\neq 0$, where ϵ is a generator of $H^5((\mathrm{SU}(4))\langle 3\rangle;\mathbb{Z})\cong\mathbb{Z}$ as above.

Proof. Since $S^3 \wedge \mathbb{R}P^2$ is 3-connected, the projection $(\mathrm{SO}(6))\langle 3\rangle \to \mathrm{SO}(6)$ induces an injection $[S^3 \wedge \mathbb{R}P^2, (\mathrm{SO}(6))\langle 3\rangle] \to [S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(6)]$ of pointed homotopy set. By Lemma 8, we know that the Samelson product $\langle \iota, \theta\rangle$ is essential, and then so is its lift $S^3 \wedge \mathbb{R}P^2 \to (\mathrm{SO}(6))\langle 3\rangle$.

Let $\tilde{\gamma} : S^3 \wedge \mathrm{SO}(6) \to (\mathrm{SO}(6))\langle 3\rangle$ be a lift of the restriction of the reduced commutator of SO(6) to $S^3 \wedge \mathrm{SO}(6) = S_{1,2}(\mathrm{SO}) \wedge \mathrm{SO}(6)$. Then we have a homotopy commutative diagram:

$$\begin{tikzcd} S^3 \wedge \mathrm{SO}(6) \arrow[r, "1\wedge\pi"] \arrow[d, "\tilde{\gamma}"'] & S^3 \wedge \mathcal{X}_{1,2}(\mathrm{SU}) \arrow[d, "\tilde{\Gamma}_{1,2}"] \\ (\mathrm{SO}(6))\langle 3\rangle \arrow[r, equal] & (\mathrm{SU}(4))\langle 3\rangle. \end{tikzcd}$$

Thus we have established that λ is essential. Now since $S^3 \wedge \mathbb{R}P^2$ is of dimension 5 and $(\mathrm{SU}(4))\langle 3\rangle$ is 4-connected, it follows from the J.H.C. Whitehead theorem that $\lambda^*(\epsilon) \neq 0$. $\square$

4. Generating variety for $\Omega_0^3 \mathrm{G}(n)$

The aim of this section is to prove that it holds for $\Omega_0^3 \mathbf{G}(d+l)$ by the maps $\Gamma_{1,l}$ and $\Gamma'_{1,l}$ in the stable range of $\Omega_0^3 \mathbf{G}(d+l)$, the generating variety argument which is analogous to single loop spaces of Lie groups in [5]. The proofs are done by a similar calculation in [15].

Theorem 10. *For $* \leqslant l$, the Pontrjagin ring $H_*(\Omega_0^3 \mathrm{SO}(4+l); \mathbb{Z}/2)$ is a polynomial ring generated by the image of $(\mathrm{ad}^3\, \Gamma'_{1,l})_* : H_*(\mathbb{H}P^{[\frac{l}{4}]}; \mathbb{Z}/2) \to H_*(\Omega_0^3 \mathrm{SO}(4+l); \mathbb{Z}/2)$.*

Proof. We first prove the case $l = \infty$. We will use the same notation for the mod 2 cohomology of $\mathrm{SO}(\infty)$ as in the proof of Proposition 5. Then, in particular, we have

$$\mathrm{Sq}^{2i-2} x_{2i-1} = x_{4i-3}, \qquad \mathrm{Sq}^{4i-3} x_{4i-1} = 0.$$

Let q and u_n be generators of $H^4(\mathbb{H}P^\infty; \mathbb{Z}/2)$ and $H^n(S^n; \mathbb{Z}/2)$ as above, respectively. Then it follows from Proposition 5 that

$$\left(\Gamma'_{1,\infty}\right)^*(x_{4i-1}) = u_3 \otimes q^{i-1}.$$

Since $\pi_1(\mathrm{SO}(\infty)) \cong \mathbb{Z}/2$, we have

$$H^*\big(\big(\mathrm{SO}(\infty)\big)\langle 1\rangle; \mathbb{Z}/2\big) = \mathbb{Z}/2\big[\pi^*(x_3), \pi^*(x_5), \pi^*(x_7), \ldots\big],$$

where $\pi : (\mathrm{SO}(\infty))\langle 1\rangle \to \mathrm{SO}(\infty)$ denotes the projection. Then, by the Borel transgression theorem, we have

$$H^*\big(\Omega_0 \mathrm{SO}(\infty); \mathbb{Z}/2\big) = \Delta(y_2, y_4, y_6, \ldots), \qquad \left(\mathrm{ad}\, \Gamma'_{1,\infty}\right)^*(y_{4i-2}) = u_2 \otimes q^{i-1}$$

and

$$y_{2i-2}^2 = \mathrm{Sq}^{2i-2} y_{2i-2} = y_{4i-4}, \qquad \mathrm{Sq}^{4i-3} y_{4i-2} = 0,$$

where y_i is the suspension of x_{i+1} and $\Delta(a_1, a_2, \ldots)$ stands for the simple system of generators $\{a_1, a_2, \ldots\}$. It is rewritten as

$$H^*\big(\Omega_0 \mathrm{SO}(\infty); \mathbb{Z}/2\big) = \mathbb{Z}/2[y_2, y_6, y_{10}, \ldots].$$

Then it follows from the Borel transgression theorem that

$$H^*\big(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2\big) = \Delta(z_1, z_5, z_9, \ldots), \qquad \left(\mathrm{ad}^2\, \Gamma'_{1,\infty}\right)^*(z_{4i-3}) = u_1 \otimes q^{i-1}$$

and

$$z_{4i-3}^2 = \mathrm{Sq}^{4i-3} z_{4i-3} = 0,$$

where z_i is the suspension of y_{i+1}. Namely, we have

$$H^*\big(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2\big) = \Lambda(z_1, z_5, z_9, \ldots).$$

Now we take the dual Hopf algebra of $H^*(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2)$ to get

$$H_*\big(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2\big) = \Lambda\big(z_1^\sharp, z_5^\sharp, z_9^\sharp, \ldots\big), \qquad \left(\mathrm{ad}^2\, \Gamma'_{1,\infty}\right)_*\big(u_1^\sharp \otimes \big(q^{i-1}\big)^\sharp\big) = z_{4i-3}^\sharp,$$

where $x^\sharp$ means the Kronecker dual of x. Since $\pi_3(\mathrm{SO}(\infty)) \cong \mathbb{Z}$, we have

$$H_*\big(\big(\Omega_0^2\,\mathrm{SO}(\infty)\big)\langle 1\rangle;\mathbb{Z}/2\big) = \Lambda(s_5, s_9, s_{13}, \ldots),$$

where s_i is defined by $\pi'_*(s_i) = z_i^\sharp$ for the projection $\pi' : (\Omega_0^2\,\mathrm{SO}(\infty))\langle 1\rangle \to \Omega_0^2\,\mathrm{SO}(\infty)$. Then, by the Borel transgression theorem, we have, for $* < l$,

$$H_*\big(\Omega_0^3\,\mathrm{SO}(\infty);\mathbb{Z}/2\big) = \mathbb{Z}/2[t_4, t_8, t_{12}, \ldots], \qquad \big(\mathrm{ad}^3\,\Gamma'_{1,\infty}\big)_*\big(\big(q^{i-1}\big)^\sharp\big) = t_{4i-4}$$

in which s_{i+1} is the transgression image of t_i, and therefore the proof is completed.

Note that the inclusion $\mathrm{SO}(4+l) \to \mathrm{SO}(\infty)$ is a $(4+l)$-equivalence. Then the inclusion $\Omega_0^3\,\mathrm{SO}(4+l) \to \Omega_0^3\,\mathrm{SO}(\infty)$ is a $(1+l)$-equivalence, and thus the theorem follows from (5). $\square$

In proving the generating variety argument for $\Omega_0^3\,\mathrm{Sp}(1+l)$, we will use:

Lemma 11. *(S. Araki and T. Kudo [1]) Let X be a simply connected homotopy associative H-space. If $H_*(X;\mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, \ldots]$ and each x_i is transgressive, then we have*

$$H_*(\Omega X;\mathbb{Z}/2) = \mathbb{Z}/2\big[y_1^0, y_1^1, \ldots, y_2^0, y_2^1, \ldots\big]$$

where y_k^l is the transgression image of $x_k^{2^l}$.

Theorem 12. *For $* \leqslant 4l+2$, the Pontrjagin ring $H_*(\Omega_0^3\,\mathrm{Sp}(1+l);\mathbb{Z}/2)$ is a polynomial ring generated by the image of $(\mathrm{ad}^3\,\Gamma_{1,l})_* : H_*(\mathbb{R}P^{4l+3};\mathbb{Z}/2) \to H_*(\Omega_0^3\,\mathrm{Sp}(1+l);\mathbb{Z}/2)$.*

Proof. We first prove the case $l=\infty$. We will use the same notation for the mod 2 cohomology of $\mathrm{Sp}(1+l)$ as in the proof of Proposition 6. Let u_n and w be generators of $H^n(S^n;\mathbb{Z}/2)$ and $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$, respectively, as well as above. Then it follows from Proposition 6 that

$$\Gamma_{1,\infty}^*(y_{4i-1}) = u_3 \otimes w^{4i-4}.$$

Now we take the dual Hopf algebra of $H^*(\mathrm{Sp}(\infty);\mathbb{Z}/2)$ so that

$$H_*\big(\mathrm{Sp}(\infty);\mathbb{Z}/2\big) = \Lambda\big(y_3^\sharp, y_7^\sharp, \ldots\big), \qquad (\Gamma_{1,\infty})_*\big(u_3^\sharp \otimes \big(w^{4i-4}\big)^\sharp\big) = y_{4i-1}^\sharp.$$

Then, by the Borel transgression theorem, we get

$$H_*\big(\Omega\,\mathrm{Sp}(\infty);\mathbb{Z}/2\big) = \mathbb{Z}/2[z_2, z_6, \ldots], \qquad (\mathrm{ad}\,\Gamma_{1,\infty})_*\big(u_2^\sharp \otimes \big(w^{4i-4}\big)^\sharp\big) = z_{4i-2}$$

in which z_i is the transgression image of y_{i+1}.

By Lemma 15 in the next section, Theorem 10 implies the map

$$\big(\mathrm{ad}^3\,\Gamma_{\infty,\infty}\big)_* : H_*\big(B\,\mathrm{Sp}(\infty);\mathbb{Z}/2\big) \to H_*\big(\Omega_0^3\,\mathrm{SO}(\infty);\mathbb{Z}/2\big)$$

is an isomorphism. Then since $B\,\mathrm{Sp}(\infty)$ and $\Omega_0^3\,\mathrm{SO}(\infty)$ are of finite type, we deduce that the map $(\mathrm{ad}^3\,\Gamma_{\infty,\infty})_{(2)} : B\,\mathrm{Sp}(\infty)_{(2)} \simeq \Omega_0^3\,\mathrm{SO}(\infty)_{(2)}$ is a homotopy equivalence, where $-_{(2)}$ means the 2-localization in the sense of Bousfield and Kan [7]. In particular, we can consider the action of the Kudo–Araki operation Q^{4i} on $q_i^\sharp \in H_*(B\,\mathrm{Sp}(\infty);\mathbb{Z}/2)$, where q_i is the mod 2 reduction of the symplectic Pontrjagin class. (See [10].) Recall that in $H^*(B\,\mathrm{Sp}(\infty);\mathbb{Z}/2)$, we have

$$q_i^\sharp = \big(q_1^\sharp\big)^i.$$

Then, in particular,

$$Q^{4i} q_i^\sharp = \big(q_i^\sharp\big)^2 = \big(q_1^\sharp\big)^{2i} = q_{2i}.$$

Since Q^{4i} commutes with the transgression, we obtain

$$Q^{4i} z_{4i-2} = z_{8i-2}.$$

Then it follows from the Nishida relation $\mathrm{Sq}_*^2 Q^s = \binom{s-2}{2} Q^{s-2} + Q^{s-1}\mathrm{Sq}_*^1$ that

$$\mathrm{Sq}_*^2 z_{8i-2} = z_{4i-2}^2,$$

where Sq_*^k denotes the dual of Sq^k. (See [10].) Since $(\mathrm{ad}\,\Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{4i-4})^\sharp) = z_{4i-2}$ and $\mathrm{Sq}_*^2(w^{4i})^\sharp = (w^{4i-2})^\sharp$, we have established

$$(\mathrm{ad}\,\Gamma_{1,\infty})_*\big(u_2^\sharp \otimes \big(w^{2i-2}\big)^\sharp\big) = z_{2i}.$$

Applying Lemma 11, we get

$$H_*\big(\Omega^2\,\mathrm{Sp}(\infty);\mathbb{Z}/2\big) = \mathbb{Z}/2[s_1, s_3, \ldots], \qquad \big(\mathrm{ad}^2\,\Gamma_{1,\infty}\big)_*\big(u_1^\sharp \otimes \big(w^{2i-2}\big)^\sharp\big) = s_{2i-1}$$

where $s_{2^m(4n-2)-1}$ is the transgression image of $z_{4n-2}^{2^m}$. Note that we can consider the operations Q^{i-1} and Q^i on z_i. By the Nishida relation $\mathrm{Sq}_*^1 Q^s = (s-1)Q^{s-1}$, we see that, for $m \geqslant 1$,

$$\mathrm{Sq}_*^1 z_{4n-2}^{2^m} = \mathrm{Sq}_*^1 Q^{2^{m-1}(4n-2)} z_{4n-2}^{2^{m-1}} = Q^{2^{m-1}(4n-2)-1} z_{4n-2}^{2^{m-1}}$$

and then

$$\mathrm{Sq}_*^1 s_{2^m(4n-2)-1} = s_{2^{m-1}(4n-2)-1}^2.$$

Thus we can deduce that

$$\big(\mathrm{ad}^2\,\Gamma_{1,\infty}\big)_*\big(u_1^\sharp \otimes \big(w^{i-1}\big)^\sharp\big) = s_i,$$

where we put $s_{2i} = s_i^2$.

Since $\pi_3(\mathrm{Sp}(\infty)) \cong \mathbb{Z}$, we have

$$H_*\big(\big(\Omega^2\,\mathrm{Sp}(\infty)\big)\langle 1\rangle;\mathbb{Z}/2\big) = \mathbb{Z}/2[\bar{s}_2, \bar{s}_3, \bar{s}_5, \ldots]$$

in which $\bar{s}_i$ is defined by $\pi_*(\bar{s}_i) = s_i$ for the projection $\pi : (\Omega^2\,\mathrm{Sp}(\infty))\langle 1\rangle \to \Omega^2\,\mathrm{Sp}(\infty)$. Then, by Lemma 11, we obtain

$$H_*\big(\Omega_0^3\,\mathrm{Sp}(\infty);\mathbb{Z}/2\big) = \mathbb{Z}/2[t_1, t_2, t_3, \ldots], \qquad \big(\mathrm{ad}^3\,\Gamma_{1,\infty}\big)_*\big(\big(w^{i-1}\big)^\sharp\big) = t_{i-1}$$

and thus the proof is done.

Since the inclusion $\mathrm{Sp}(1+l) \to \mathrm{Sp}(\infty)$ is a $(4l+6)$-equivalence, the inclusion $\Omega_0^3\,\mathrm{Sp}(1+l) \to \Omega_0^3\,\mathrm{Sp}(\infty)$ is a $(4l+3)$-equivalence. Therefore the proof is completed by (2). □

We next consider the case $\mathbf{G} = \mathrm{SU}$. Only in this case, we will use a result related with Bott periodicity which is an easy consequence of [19].

Lemma 13. *Let a be a generator of $H^2(\Omega_0^3\,\mathrm{SU}(\infty);\mathbb{Z}) \cong \mathbb{Z}$. Then the integral homology of $\Omega_0^3\,\mathrm{SU}(\infty)$ is*

$$H_*\big(\Omega_0^3\,\mathrm{SU}(\infty);\mathbb{Z}\big) = \mathbb{Z}[b_2, b_4, \ldots], \qquad b_{2i} = \big(a^i\big)^\sharp.$$

Theorem 14. *For $* \leqslant 2l$, the Pontrjagin ring $H_*(\Omega_0^3\,\mathrm{SU}(\infty);\mathbb{Z})$ is a polynomial ring generated by the image of $(\mathrm{ad}^3\,\tilde{\Gamma}_{1,l})_* : H_*(\mathcal{X}_{1,l}(\mathrm{SU});\mathbb{Z}) \to H_*(\Omega_0^3\,\mathrm{SU}(2+l);\mathbb{Z})$.*

Proof. The case $l = \infty$ follows from Proposition 7 and Lemma 13. One can easily verify that the inclusion $\Omega_0^3\,\mathrm{SU}(2+l) \to \Omega_0^3\,\mathrm{SU}(\infty)$ and the natural map $\mathcal{X}_{1,l}(\mathrm{SU}) \to \mathcal{X}_{1,\infty}(\mathrm{SU})$ are $(2l+4)$-equivalences. Thus the theorem follows from (2). □

5. Bott periodicity

In this section, we prove that the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ is a homotopy equivalence. Notice here that we have not used any result concerning real and symplectic Bott periodicity. We have only used the result of Toda [19] to get the ring structure of $\Omega_0^3\,\mathrm{SU}(\infty)$ in the last section. Then our result provides a new proof for real and symplectic Bott periodicity.

We start with an easy algebraic lemma. Let V be a graded free module over a PID. As usual, we will call V of finite type if, in each dimension, V is finitely generated. We will denote the free commutative graded algebra generated by V by ΛV. Then we can easily see:

Lemma 15. *(Kono and Tokunaga [16]) Let V and W be of finite type graded free modules over a PID R such that $V \cong W$, and let U be a graded module over R. Given a graded algebra map $f : \Lambda V \to \Lambda W$ and a graded module map $g : U \to \Lambda V$. If the image of $f \circ g : U \to \Lambda W$ generates ΛW, then f is an isomorphism.*

Now we prove our main theorem.

Theorem 16. *The map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ is a homotopy equivalence.*

Proof. We first prove the case $\mathbf{G} = \mathrm{SU}$. By Lemma 4, Theorem 14 and Lemma 15 together with the homotopy commutative diagram (6), we see that the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\mathrm{U}(\infty) \to \Omega_0^3\,\mathrm{SU}(\infty)$ induces an isomorphism in the integral homology. Then, by the J.H.C. Whitehead theorem, we obtain that $\mathrm{ad}^3\,\Gamma_{\infty,\infty}$ is a homotopy equivalence. Thus, in particular, from $\pi_*(B\mathrm{U}(2))$ is for $* \leqslant 4$, we deduce:

$$\pi_*\big(B\mathrm{U}(\infty)\big) \cong \begin{cases} \mathbb{Z}, & * = 2, 4, \ldots, \\ 0, & * = 1, 3, \ldots. \end{cases} \tag{9}$$

Note here that we do not need to use Bott periodicity of $B\mathrm{U}(\infty)$.

We next consider the case $\mathbf{G} = \mathrm{SO}$. We may assume $\Gamma'_{1,\infty} = \Gamma_{1,\infty}$ as noted above. Then it follows from Lemma 4, Theorem 10, Lemma 15 and (6) that the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\,\mathrm{Sp}(\infty) \to \Omega_0^3\,\mathrm{SO}(\infty)$ induces an isomorphism in the mod 2 homology. On the other hand, we have $\mathbf{qc}' = 1 : B\,\mathrm{Sp}(\infty) \to B\,\mathrm{Sp}(\infty)$ and $\mathbf{rc} = 2 : B\,\mathrm{SO}(\infty) \to B\,\mathrm{SO}(\infty)$. Then it follows from (9) that the homotopy groups of $B\,\mathrm{Sp}(\infty)$ and $\Omega_0^3\,\mathrm{SO}(\infty)$ are odd torsion free. Then, by considering the rational cohomology of $B\,\mathrm{Sp}(\infty)$ and $B\,\mathrm{SO}(\infty)$, we obtain

$$\pi_*\big(B\,\mathrm{Sp}(\infty)\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \begin{cases} \mathbb{Z}[\frac{1}{2}], & * = 4, 8, \ldots, \\ 0, & * \neq 4, 8, \ldots \end{cases}$$

and

$$\pi_*\big(\Omega_0^3\,\mathrm{SO}(\infty)\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \begin{cases} \mathbb{Z}[\frac{1}{2}], & * = 4, 8, \ldots, \\ 0, & * \neq 4, 8, \ldots. \end{cases}$$

This implies that the maps $\mathbf{c}'_* : \pi_*(B\,\mathrm{Sp}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(B\mathrm{U}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ and $\mathbf{c}_* : \pi_*(\Omega_0^3\,\mathrm{SO}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega_0^3\,\mathrm{SU}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ are split monomorphisms. Thus since $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\mathrm{U}(\infty) \to \Omega_0^3\,\mathrm{SU}(\infty)$ is a homotopy equivalence as above, the map $(\mathrm{ad}^3\,\Gamma_{\infty,\infty})_* : \pi_*(B\,\mathrm{Sp}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega_0^3\,\mathrm{SO}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism by Proposition 3. On the other hand, we can apply Lemma 15 to the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\,\mathrm{Sp}(\infty) \to \Omega_0^3\,\mathrm{SO}(\infty)$ in the mod 2 homology by Lemma 4 and Theorem 10. Then we obtain the map $\mathrm{ad}^3\,\Gamma_{\infty,\infty} : B\,\mathrm{Sp}(\infty) \to \Omega_0^3\,\mathrm{SO}(\infty)$ induces an isomorphism in the mod 2 homology. Summarizing, we have established that this map is a homology equivalence and therefore by a generalized J.H.C. Whitehead theorem [11], the proof is completed.

The case $\mathbf{G} = \mathrm{Sp}$ is quite similar to the case $\mathbf{G} = \mathrm{SO}$. $\square$

Corollary 17. *Let $d_{k,l} = \min\{2k+1, 2l+1\}, \min\{k, 4l+3\}, \min\{4k+3, l\}$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then the map $\mathrm{ad}^3\,\Gamma_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \Omega_0^3\mathbf{G}(dk+l)$ is a $d_{k,l}$-equivalence.*

Proof. Let $a_k = 2k+1, k, 4k+3$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then it is easy to see that the projection $B\mathbf{H}(k) \to B\mathbf{H}(\infty)$ is an a_k-equivalence. By definition, there is a principal bundles

$$\mathbf{H}(k) \to \mathbf{G}(dk+l)/\mathbf{G}(l) \to \mathcal{X}_{k,l(\mathbf{G})}$$

for $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$ and

$$\mathrm{U}(k) \to \mathrm{U}(2k+l)/\mathrm{U}(l) \to \mathcal{X}_{k,l}(\mathrm{SU}).$$

Let $b_{k,l} = 2l+1, 4l+3, l$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then it follows from the above principal bundles that the composite of the inclusion $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{k,\infty}(\mathbf{G})$ and the homotopy equivalence $\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$ is a $b_{k,l}$-equivalence. Let $c_{k,l} = 4k+2l-3$, $4k+4l-1$, $4k+l-4$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then the inclusion $\Omega_0^3\mathbf{G}(dk+l) \to \Omega_0^3\mathbf{G}(\infty)$ is a $c_{k,l}$-equivalence. Now let us consider a homotopy commutative diagram:

$$\begin{tikzcd}
\mathcal{X}_{k,l}(\mathbf{G}) \arrow[r] \arrow[d, "\mathrm{ad}^3\,\Gamma_{k,l}"'] & B\mathbf{H}(k) \arrow[r] \arrow[d, "\mathrm{ad}^3\,\Gamma_{k,\infty}"'] & B\mathbf{H}(\infty) \arrow[d, "\mathrm{ad}^3\,\Gamma_{\infty,\infty}"'] \\
\Omega_0^3\mathbf{G}(dk+l) \arrow[r] & \Omega_0^3\mathbf{G}(\infty) \arrow[r, equal] & \Omega_0^3\mathbf{G}(\infty).
\end{tikzcd}$$

Then it follows from Theorem 16 that the map $\mathrm{ad}^3\,\Gamma_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \Omega_0^3\mathbf{G}(dk+l)$ is a $\min\{a_k, b_{k,l}, c_{k,l}\}$-equivalence. Thus the proof is completed. $\square$

6. Applications to instanton moduli spaces

In this section, we give applications of the results obtained so far to the homotopy types of instanton moduli spaces $\mathcal{M}_k(G)$.

Recall from Lemma 1 that the map $\Gamma_{k,l} : S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G}) \to \mathbf{G}(dk+l)$ was constructed from the moduli space of $\mathbf{G}(dk+l)$-instantons on S^4. In particular, $\mathcal{X}_{k,l}(\mathbf{G})$ is a subspace of $\mathcal{M}_k(\mathbf{G}(dk+l))$. We denote the inclusion $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(dk+l))$ by $i_{k,l}$. Then, by definition, we have

$$\mathrm{ad}^3\,\Gamma_{k,l} = j_{k,l} \circ i_{k,l} \tag{10}$$

where $j_{k,l} : \mathcal{M}_k(\mathbf{G}(dk+l)) \to \Omega_0^3\mathbf{G}(dk+l)$ is the inclusion. We also have a commutative diagram:

$$\begin{tikzcd}
\mathcal{X}_{k,l}(\mathbf{G}) \arrow[r] \arrow[d, "i_{k,l}"'] & \mathcal{X}_{k,l+1}(\mathbf{G}) \arrow[d, "i_{k,l+1}"] \\
\mathcal{M}_k(\mathbf{G}(dk+l)) \arrow[r] & \mathcal{M}_k(\mathbf{G}(dk+l+1)).
\end{tikzcd}$$

Here the horizontal arrows are induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$. Then we have a map

$$\operatorname*{colim}_l i_{k,l} : \mathcal{X}_{k,\infty}(\mathbf{G}) \to \operatorname*{colim}_l \mathcal{M}_k\big(\mathbf{G}(dk+l)\big)$$

which we denote by $i_{k,\infty} : \mathcal{X}_{k,\infty}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(\infty))$.

Proposition 18. *The map $i_{k,\infty}$ is a homotopy equivalence.*

Proof. We first prove the case $\mathbf{G} = \mathrm{SU}, \mathrm{SO}$. Recall from [17] that there is a homotopy equivalence $\mathcal{M}_k(\mathbf{G}(\infty)) \simeq B\mathbf{H}(k)$. On the other hand, we know that $\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$. Then we have $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z}) \cong H^*(\mathcal{M}_k(\mathbf{G}(\infty));\mathbb{Z})$ as abstract rings. Note that $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z})$ is a polynomial ring. By Corollary 17, we see that $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z})$ is generated by $\mathrm{Im}(\mathrm{ad}^3\,\Gamma_{k,\infty})^*$. Therefore, by Lemma 15 and (10), the proof is completed.

We next prove the case $\mathbf{G} = \mathrm{Sp}$. By Corollary 17, the map $(\mathrm{ad}^3\,\Gamma_{k,\infty})^* : H^*(\Omega_0^3\,\mathrm{Sp}(\infty)\mathbb{Z}/2) \to H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z}/2)$ is an isomorphism for $* \leqslant k$. Then, in particular, it follows from (10) that the map $(i_{k,\infty})_* : \pi_1(\mathcal{X}_{k,\infty}(\mathbf{G})) \to \pi_1(\mathcal{M}_k(\mathbf{G}(\infty)))$ is an isomorphism, where both $\pi_1\mathcal{X}_{k,\infty}(\mathbf{G})$ and $\pi_1(\mathcal{M}_k(\mathbf{G}(\infty)))$ are isomorphic to $\mathbb{Z}/2$. Since both $\mathcal{X}_{k,\infty}\langle 1\rangle$ and $\mathcal{M}_k(\mathbf{G}(\infty))\langle 1\rangle$ have the homotopy type of $B\,\mathrm{SO}(k)$, we can see the map $(i_{k,\infty})\langle 1\rangle : \mathcal{X}_{k,\infty}\langle 1\rangle \to \mathcal{M}_k(\mathbf{G}(\infty))\langle 1\rangle$ induces an isomorphisms in the cohomology with the coefficients $\mathbb{Z}/2$ and $\mathbb{Z}[\frac{1}{2}]$ quite analogously to the above case. Then the map $(i_{k,\infty})\langle 1\rangle$ is a homology equivalence, and hence a homotopy equivalence. Therefore the map $i_{k,\infty}$ is a homotopy equivalence. □

We estimate a range that the map $i_{k,l} : \mathcal{X}_{k,l}(\mathrm{SU}) \to \mathcal{M}_k(\mathrm{SU}(dk+l))$ is a homotopy equivalence.

Theorem 19. *The map $i_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(dk+l))$ is a $\min\{2k+1, 2l+1\}$-equivalence.*

Proof. In [14], it is shown that the map $\mathcal{M}_k(\mathbf{G}(dk+l)) \to \mathcal{M}_k(\mathbf{G}(\infty))$ induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$ is a $(2k+1)$-equivalence. On the other hand, the map $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{k,\infty}(\mathbf{G})$ induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$ is a $(2l+1)$-equivalence as is seen in the proof of Corollary 17. Then the theorem follows from (2) and Proposition 18. □

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